# CONVECTIVE INSTABILITY OF HORIZONTAL FLUID LAYERS BOUND BY THERMAL INTERACTION

*PMM Vol. 32, No. 3, 1968, pp. 478-481*

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*(Received January 28, 1968)*

The problem of convection onset in a system consisting of two infinite flat horizontal layers of fluid separated by a solid heat conducting mass is examined. Critical values of the Rayleigh number defining the equilibrium stability in terms of the distance between layers, and of the fluid and solid mass thermal conductivity ratio is established. It is shown that the thermal interaction of layers via the heat conducting interlayer leads to a lowering of stability.

**1.** We shall consider two infinite flat layers of identical thickness $h$ of a fluid surrounded by a solid heat-conducting mass. The distance between the two fluid layers inner boundaries is $2d$ (Fig. 1). It is assumed that the physical parameters of the fluid filling the two layers are identical, and that the thermal conductivity of the solid interlayer and that of the outer boundary masses are equal.

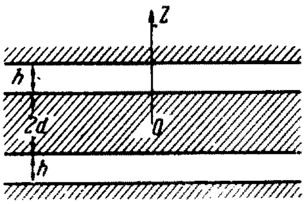


Fig. 1

Under stabilized conditions the fluid velocity is zero, and the temperature gradients in each of the fluid layers, and in the solid mass are vertical and constant.

In the equilibrium state the temperature gradients $A$ in the fluid and $A_m$ in the solid mass are bound by the vertical heat flux continuity condition

$$\varkappa A = \varkappa_m A_m \tag{1.1}$$

where $\kappa$ and $\kappa_m$ are the values of the fluid and solid mass thermal conductivity.

Equations of small perturbations may be derived in the usual manner from convection equations with the perturbation monotony principal taken into account. Selecting the following units: for distance – the fluid layer thickness $h$, for velocity $\chi/h$ ($\chi$ is the fluid thermal diffusivity), for temperature $Ah$, and for pressure $\rho\nu\chi/h^2$ ($\rho$ and $\nu$ are respectively the fluid density and kinematic viscosity), we obtain the following dimensionless equations for the stationary perturbations at the limit of stability:

$$\nabla p = \Delta v + RT\gamma \qquad \left(R = \frac{g\beta A h^4}{\nu\chi}\right) \tag{1.2}$$

$$\operatorname{div} v = 0, \qquad \Delta T = -v_z, \qquad \Delta T_m = 0 \tag{1.3}$$

Here $v$ is the fluid velocity, $T$ and $T_m$ are temperature perturbations of the fluid and solid mass respectively, $p$ is the pressure perturbation, and $\gamma$ is a unit vector directed upwards. The Rayleigh number $R$ appearing in system (1.2), (1.3) is defined by the temperature gradient in the fluid, the layer thickness and the fluid parameters.

Eliminating from system (1.2), (1.3) the pressure and the velocity horizontal components, and assuming that the dependence of normal perturbations on horizontal coordinates is expressed by $\exp i\,(k_1 x + k_2 y)$, we obtain equations for the amplitude of vertical velocity $v\,(z)$, and of the amplitude of temperature perturbation $\theta\,(z)$ and $\theta_m(z)$

$$v^{IV} - 2k^2 v'' + k^4 v = k^2 R\theta \qquad (k^2 = k_1^2 + k_2^2) \tag{1.4}$$

$$\theta'' - k^2\theta = -v, \qquad \theta_m'' - k^2\theta_m = 0 \tag{1.5}$$

At the fluid-solid boundary all velocity components vanish, while the temperature and the heat flux are continuous. From this we derive the boundary conditions which must be fulfilled at the inner and outer boundaries of the fluid layers

$$v = v' = 0, \quad \theta = \theta_m, \quad \lambda\theta' = \theta_m' \quad \text{for} \quad z = z_1,\ z = z_2 \quad (\lambda = \varkappa / \varkappa_m) \tag{1.6}$$

Here $z_1$ and $z_2$ are the inner and outer boundaries of the layers, and $\kappa$ and $\kappa_m$ are the values of the thermal conductivity of fluid and solid.

Temperature perturbations in the outer solids vanish at infinity

$$\theta_m \to 0 \quad \text{for} \quad z \to \pm\infty \tag{1.7}$$

The boundary value problem (1.4) to (1.7) defines the critical Rayleigh numbers $R$, as well as the critical perturbations corresponding to these.

Vertical velocity and temperature perturbations, even with respect to $z$, obviously correspond to the basic instability level. Hence, when considering the latter, it will be sufficient to solve the problem in the area of $z > 0$ on the assumption that the temperature perturbation in the solid interlayer is an even function of $z$.

2. An effective approximate solution determining the basic instability level may be obtain with the aid of the Galerkin method. For this we approximate the amplitude of vertical velocity $v\,(z)$ in conformity with boundary conditions (1.6) in the following manner

$$v = c\,(z - z_1)^2\,(z_2 - z)^2 \tag{2.1}$$

Here $c$ is a constant coefficient which by virtue of the problem homogeneity remains arbitrary, and in accordance with the normalization condition is in the following assumed to be equal to unity.

The corresponding temperature distribution $\theta$ in the fluid, $\theta_{m1}$ in the solid interlayer, and $\theta_{m2}$ in the outer solid mass will be found by solving the thermal conductivity Eq. (1.5) for a given velocity distribution (2.1).

Taking into account (1.7) the solution of the heat-conductivity equation for the outer solid mass is of the form

$$\theta_{m2} = Ae^{-kz} \qquad (z \geqslant z_2) \tag{2.2}$$

Temperature distribution in the solid interlayer is defined by the even function

$$\theta_{m1} = B\,\mathrm{ch}kz \qquad (z \leqslant z_1) \tag{2.3}$$

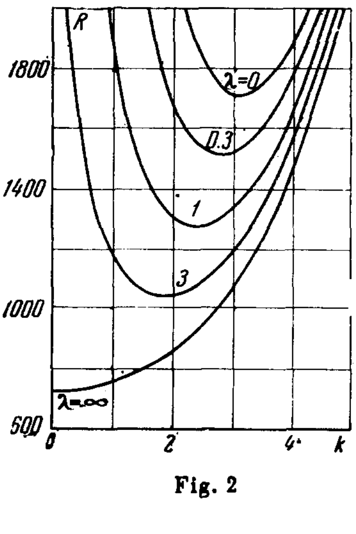

Fig. 2

Solution of the thermal conductivity equation in the area of $z_1 \leqslant z \leqslant z_2$ yields the temperature distribution in the fluid layer

$$\theta = C_1\,\mathrm{ch}ku + C_2\,\mathrm{sh}ku + k^{-6}\,[k^4u^2\,(1-u)^2 + 2k^2\,(6u^2 - 6u + 1) + 24] \qquad (u = z - z_1) \tag{2.4}$$

Constants of integration $A$, $B$, $C_1$, and $C_2$ are derived from the temperature and heat flux continuity conditions at the inner ($z = z_1$) and outer ($z = z_2 = z_1 + 1$) fluid boundaries. We shall adduce constants $C_1$ and $C_2$ which define temperature perturbations in the fluid only

$$C_1 = -\delta^{-1}[a(\mathrm{sh}k + \lambda\,\mathrm{ch}k)\,\mathrm{th}kz_1 + \lambda(a + b\,\mathrm{sh}k) + \lambda^2 b(1 + \mathrm{ch}k)]$$
$$C_2 = -\delta^{-1}[a(1 - \mathrm{ch}k)\,\mathrm{th}kz_1 + \lambda(b - a\mathrm{sh}k)\,\mathrm{th}kz_1 - b\lambda(\mathrm{ch}k + \lambda\mathrm{sh}\,k)] \quad (2.5)$$
$$\delta = (\mathrm{sh}k + \lambda\,\mathrm{ch}k)\,\mathrm{th}\,kz_1 + \lambda(\mathrm{ch}k + \lambda\mathrm{sh}\,k), \qquad a = 2k^{-6}(12 + k^2), \qquad b = 12k^{-5}$$

Substituting into the Navier-Stokes Eq. (1.4) the distribution of velocity (2.1) and of temperature (2.4), multiplying by function $v(z)$ and integrating with respect to $z$ from $z_1$ to $z_2$, we obtain a relation from which the critical value of the Rayleigh number is found

$$R = \frac{(f_1 + \lambda f_2)\,\mathrm{th}\,kz_1 + \lambda f_2 + \lambda^2 f_1}{(f_3 + \lambda f_4)\,\mathrm{th}\,kz_1 + \lambda f_4 + \lambda^2 f_5} \quad (2.6)$$

Here functions $f_i$ depend only on the wave number $k$

$$f_1 = (504 + 24k^2 + k^4)k^9, \qquad f_2 = (504 + 24k^2 + k^4)k^9\,\mathrm{cth}\,k$$
$$f_3 = (504 - 12k^2 + k^4)k^5 + 5040(12 + k^2)[6k - (12 + k^2)\,\mathrm{th}\,{}^1/_2\,k]$$
$$f_4 = (504 - 12k^2 + k^4)k^5\,\mathrm{cth}\,k + 2520[12k(12 + k^2)\,\mathrm{csch}\,k - (144 - 12k^2 + k^4)] \quad (2.7)$$
$$f_5 = (504 - 12k^2 + k^4)k^5 + 30240\,k[6k\,\mathrm{cth}\,k/2 - (12 + k^2)]$$

Formula (2.6) defines the Rayleigh number critical value in terms of three parameters, viz., the perturbation wave number $k$, the fluid and solid mass conductivity ratio $\lambda$ and the geometric parameter $z_1$ ($2z_1$ is the solid interlayer thickness in units of the fluid layer thickness).

3. The critical Rayleigh number which determines the convection onset in the limit case of a single horizontal fluid layer bounded at its lower and upper surfaces by solid masses of an arbitrary thermal conductivity may, first of all be derived from the general Formula (2.6). This extension of the known Rayleigh problem to the case of solid boundaries of arbitrary thermal conductivity was undertaken in papers [1 and 2].

In order to proceed with the case of a single layer it is necessary to pass in (2.6) to the limit $z_1 \to \infty$, as in the presence of an infinitely thick interlayer any interaction between fluid layers is absent, and the two layers become independent.

Assuming $z_1 \to \infty$ we obtain from (2.6)

$$R = \frac{(\lambda + \mathrm{cth}\,{}^1/_2 k)\,f_1}{\lambda f_5 + (2f_4 - f_5\,\mathrm{th}\,{}^1/_2 k)} \quad (3.1)$$

Neutral curves $R(k)$ calculated from Formula (3.1) for various values of the fluid and solid mass thermal conductivity ratios are shown on Fig. 2. The minimal critical number $R_m$ decreases with increasing $\lambda$ from the value $R_m = 1708$ for $\lambda = 0$ (a solid of infinitely great thermal conductivity) to the value $R_m = 720$ for $\lambda = \infty$ (a nonheat-conducting solid). These limit values of $R_m$ practically coincide with the values derived in [2] from the exact characteristical relationship.

We shall now pass to the consideration of results ensuing from the general Formula (2.6).

For each fixed value of parameter $z_1$ defining the distance between the fluid layers, and a fixed thermal conductivity ratio $\lambda$ we may plot with the aid of (2.6) a neutral curve $R(k)$, and after minimization derive values of $R_m$.

The dependence of the minimal value of the critical number $R_m$ on $z_1$ calculates by this method for several values of parameter $\lambda$ are shown on Fig. 3. The whole family of curves lies between lines $R_m = 1708$ and $R_m = 720$ which correspond to the limit cases of a solid mass of infinitely great thermal conductivity ($\lambda = 0$) and of a nonconductive

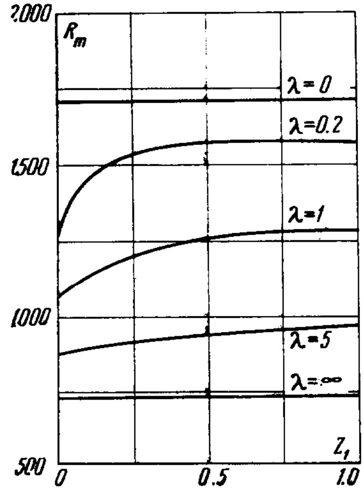


Fig. 3

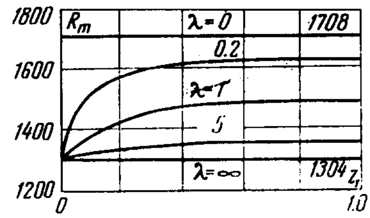


Fig. 4

solid ($\lambda = \infty$). In these limit cases the minimal critical Rayleigh number is evidently independent of the interlayer thickness. For any intermediate values of $\lambda$ the critical numbers $R_m$ increase with increasing $z_1$, and with $z_1 \to \infty$ tend to their limit values depending on $\lambda$ (as already noted these limit values coincide with the critical Rayleigh numbers for a single layer).

It should be noted that with increasing $z_1$ the critical numbers $R_m$ very rapidly reach their limit values. Critical values corresponding to noninteracting layers are practically reached when $z_1 > 0.5$, i.e. when the thickness of the solid interlayer equals that of the fluid layer.

For a given $\lambda$ the critical wave number $k_m$ increases slowly with increasing $z_1$. Thus, for $\lambda = 1$ at increasing $z_1$ from 0 to $\infty$, $k_m$ varies from 2.0 to 2.4.

Thus, with increasing distance between layers (i.e. with a weakening of their thermal bond) the convection stability increases and the critical wavelength (the horizontal dimension of the Bénard cell) decreases. We may note in connection with this that in the problem of convection stability of two vertical flat fluid layers, previously considered in [3] the opposite takes place, namely a decrease of the Rayleigh number lower critical value occurs when the distance between layers is increased. This is explained by the presence of not only thermal interaction of fluid layers through the solid interlayer, but also by the hydrodynamic effects consequent to the flow of fluid from one vertical channel to another (due to this effect distant vertical fluid layers remain hydrodynamically bound).

4. The problem considered above relates to the case of a solid interlayer having the same thermal conductivity as the outside solid masses. An extension to the case of different thermal conductivities of the interlayer and of the outside solids results in an unwieldy formula for the critical Rayleigh number. We shall only adduce the results related to the case of arbitrary thermal conductivity of the interlayer and an infinitely great thermal conductivity of the outer solid masses (temperature perturbations at the fluid outer boundaries disappear in this case). Retaining approximation (2.1) we obtain instead of (2.6)

$$R = \frac{f_1 \operatorname{th} kz_1 + \lambda f_2}{f_3 \operatorname{th} kz_1 + \lambda f_4} \qquad (4.1)$$

Here $\lambda$ is now the fluid and interlayer thermal conductivity ratio, while functions $f_i$ are defined as previously by (2.7).

The dependence of minimal $R_m(z_1)$ are shown on Fig. 4 for various values of $\lambda$. As in the previously considered case we have here an increasing stability with increasing distance between layers. For $\lambda = 0$ and $\lambda = \infty$ the critical Rayleigh numbers are independent of the distance between layers. All curves of this family originate for $z_1 = 0$ at one point (Fig. 4) (the thermal conductivity of a zero thickness interlayer is immaterial).

At the limit when $z_1 \to \infty$ we have the problem of stability of a single horizontal layer bounded on one side by a perfectly heat conducting solid mass, and on the other by a solid of an arbitrary thermal conductivity. (This problem was considered in [1]). For $z_1 \to \infty$ we obtain from Formula (4.1)

$$R = \frac{f_1 + \lambda f_2}{f_3 + \lambda f_4} \tag{4.2}$$

For $\lambda = \infty$ (one of the solids is perfectly heat-conducting, and the other nonheat-conducting) the minimal critical Rayleigh number is $R_m = 1304$ which is very close to 1296, the value found in [1].

Authors wish to express their thanks to V. Rusakov, N. Baiadina and D. Shvartsblat for their help in carrying out computations.

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Translated by J.J.D.